Pergamon

J. Appl. Maths Mechs, Vol. 58, No. 1, pp. 109–116, 1994

0021–8928/94 $24.00+0.00

0021–8928(94)E0030-E

SOMIGLIANA'S FORMULAE FOR SOLVING THE ELASTODYNAMIC EQUATIONS FOR TRAVELLING LOADS†

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(*Received* 6 *February* 1993)

Regular representations of the Somigliana formulae are constructed to solve elastodynamic equations for travelling loads. For supersonic velocities a near-front regularization of the integrand functions is proposed.

One of the methods of constructing the boundary integral equations in boundary-value problems in the theory of elasticity is based on obtaining Somigliana-type formulae expressing displacements inside the domain in terms of boundary values of stresses and displacements and fundamental solution tensors. These have been previously constructed [1] for the equations of the theory of elasticity of an isotropic body in the case of time-independent travelling loads.

1. STATEMENT OF THE PROBLEM

Consider the class of self-similar solutions

$$\mathbf{u} = \mathbf{u}(x_1, x_2, x_3 - ct) \tag{1.1}$$

of the equations of motion of a linearly-elastic homogeneous medium [2]

$$\sigma_{ij,j} - \rho u_{i,tt} + G_i = 0 \tag{1.2}$$

$$\sigma_{ij} = C_{ijkl} u_{k,l} \tag{1.3}$$

where σ_{ij} and u_j are the Lagrangian Cartesian components of the displacements and stress tensor, and C_{ijkl} is the tensor of the constants of elasticity, which in the isotropic case has the form

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{1.4}$$

If the vector $\{G_i\}$ has the structure of (1.1), it is natural to seek a solution in analogous form. Here the symbol after the comma denotes differentiation with respect to the appropriate coordinate or t, and repeated indices i, j, k, l imply summation from 1 to 3.

†*Prikl. Mat. Mekh.* Vol. 58, No. 1, pp. 103–109, 1994.

We write Eqs (1.2) in the moving system of coordinates $(x_1', x_2', x_3') = (x_1, x_2, x_3 - ct)$ taking (1.3) and (1.4) into account

$$A_{ij}\left(\frac{\partial}{\partial x'}\right)u_j = (c_1^2 - c_2^2)u_{j,ij} + c_2^2 \Delta u_i - c^2 u_{i,33} + G_i = 0 \tag{1.5}$$

where c_1 and c_2 are the velocities of bulk and shear waves in the elastic medium and Δ is the Laplace operator in R_3. We will now drop the primes in x_j'.

Let S be some sufficiently smooth closed cylindrical surface whose generator is parallel to the x_3 axis of the Cartesian system of coordinates (x_1, x_2, x_3), and let $\mathbf{n} = (n_1, n_2, n_3)$ be the unit vector of the outward normal to S. The surface S separates the domains S^- and S^+ in R_3. We shall assume that $\mathbf{u}$ is a solution to Eq. (1.5) defined in $S^- + S$, and that it is known that

$$u_j(\mathbf{x}) = u_j^s(\mathbf{x}), \quad \sigma_{ij}(\mathbf{x}) = n_j(\mathbf{x}) = p_i(\mathbf{x}), \quad \mathbf{x} \in S \tag{1.6}$$

It is required to find $\mathbf{u}(\mathbf{x})$ in the domain $\mathbf{x} \in S^-$, i.e. to construct an analogue of Somigliana's formula for travelling loads.

2. FUNDAMENTAL SOLUTIONS

It has been shown [1] that the generalized solution $\mathbf{u} = \mathbf{u}^*(\mathbf{x})H_s^-(\mathbf{x})$ of Eq. (1.5) can be represented in the form

$$\begin{aligned}\rho u_i(\mathbf{x}) = U_{ij}^* * p_j(\mathbf{x})\delta_s(\mathbf{x}) + \lambda u_k = n_k = (\mathbf{x})\delta_s(\mathbf{x}) * U_{im,m}^* + \\ + \mu(u_j n_m + u_m n_j)\delta_s(\mathbf{x}) * U_{ij,m}^* + U_{ij}^* * G_j H_s^-(\mathbf{x})\end{aligned} \tag{2.1}$$

$$H_s^-(\mathbf{x}) = \begin{cases} 1, & \mathbf{x} \in S^-, \\ 1/2, & \mathbf{x} \in S, \\ 0, & \mathbf{x} \in S^+ \end{cases}$$

where $\delta_s(\mathbf{x})$ is a simple layer on S [3], $H_s^-(\mathbf{x})$ is the characteristic function of the set S^-, and U_{ij}^* is Green's tensor of the equations of motion (1.2) corresponding to a concentrated body force $G_i = -\delta_{(i)j}\delta(\mathbf{x})$ (δ_{ij} being the Kronecker delta and $\delta(\mathbf{x})$ the delta-function)

$$U_{ij}^*(\mathbf{x}) = c_2^{-2}\delta_{ij} f_2(r, x_3) + c^{-2}(f_{1ij}(r, x_3) - f_{2ij}(r, x_3)) \tag{2.2}$$

Here

$$4\pi f_j(r, x_3) = \begin{cases} (m_j^2 r^2 + x_3^2)^{-1/2}, & c < c_j \\ -2\delta(x_3)\ln r, & c = c_j \\ 2H(-x_3 - m_j r)(x_3^2 - m_j^2 r^2)^{-1/2}, & c > c_j \end{cases}$$

$$4\pi f_{kij} = -\frac{1}{V_k^+}\left(\frac{x_3}{r}(\delta_{i3} r_{,j} + \delta_{j3} r_{,i}) - \frac{x_3^2}{r^2} r_{,i} r_{,j} - \delta_{i3}\delta_{j3}\right) + \frac{V_k^+}{r^2}(\delta_{ij}\ni_{[i]3} - r_{,i} r_{,j}), \quad c < c_k$$

$$2\pi f_{kij} = H(-x_3 - m_k r)\left(\frac{1}{V_k^-}\left(\frac{x_3}{r}(\delta_{i3} r_{,j} + \delta_{j3} r_{,i} - \frac{x_3^2}{r^2} r_{,i} r_{,j} - \delta_{i3}\delta_{j3}\right) - \right.$$

$$-\frac{V_k^-}{r^2}(\delta_{ij}\ni_{[i]3}-r_{,i}r_{,j})\Bigg), \quad c>c_k$$

$$2\pi f_{kij}=-\delta(x_3)\delta_{i3}\delta_{j3}\ln r-\frac{H(-x_3)}{r}(\delta_{i3}r_{,j}+\delta_{j3}r_{,i})-\frac{H(-x_3)x_3}{r^2}(\delta_{ij}\ni_{[i]3}-2r_{,i}r_{,j}), \quad c=c_k$$

$$r=\sqrt{x_1^2+x_2^2}, \quad r_{,j}=\frac{\partial r}{\partial x_j}, \ni_{ij}=1-\delta_{ij}$$

$$m_k=\sqrt{|1-M_k^2|}, \quad V_k^{\pm}=\sqrt{x_3^2\pm m_k^2 r^2}$$

$H(x)$ is the Heaviside step function and $M_k=c/c_k$ is the Mach number (with no summation over subscripts in square brackets).

It is convenient to separate the bulk and shear components of the tensor U_{ij}^*

$$U_{ij}^*=U_{ij1}^*+U_{ij2}^* \tag{2.3}$$
$$U_{ij1}^*=c^{-2}f_{1ij}(r,x_3), \quad U_{ij2}^*=c_2^{-2}\delta_{ij}f_2(r,x_3)-c^{-2}f_{2ij}(r,x_3)$$

Using Hooke's law (1.3), (1.4), we introduce the fundamental stress tensors

$$S_{ijk}^*=\lambda U_{mk,m}^*\delta_{ij}+\mu(U_{ik,j}^*+U_{jk,i}^*) \tag{2.4}$$
$$\Gamma_{ik}^*(\mathbf{x},\mathbf{n})=S_{ijk}^*(\mathbf{x})n_j$$

It follows from (1.2) that

$$S_{ijk,i}^*=\rho c^2U_{ik,33}^*-\rho\delta_{jk}\delta(\mathbf{x})$$

from which by a convolution with $H_F^-(\mathbf{x})\delta_{km}$ we obtain

$$S_{ijm}^* * \nu_i(\mathbf{x})\delta_F(\mathbf{x})=\rho c^2(U_{jm}^* * \nu_3(\mathbf{x})\delta_F(\mathbf{x}))_{,3}-\rho\delta_{jm}H_F^-(\mathbf{x}) \tag{2.5}$$

Here F is an arbitrary sufficiently smooth surface in R_3 bounding the set F^-, and $\nu(\mathbf{x})$ is the unit vector of the outward normal to F.

It is convenient to introduce the transposed tensor

$$T_{ij}^*(\mathbf{x},\mathbf{n})=\Gamma_{ji}^*(\mathbf{x},\mathbf{n}) \tag{2.6}$$

which allows one to write (2.5) in integral form

$$\int_F T_{ij}(\mathbf{y},\mathbf{x},\nu(\mathbf{y}))ds(\mathbf{y})=\rho\delta_{ij}H_F^-(\mathbf{x})-\rho c^2\frac{\partial}{\partial x_3}\int_F U_{ij}(\mathbf{x},\mathbf{y})\nu_3(\mathbf{y})ds(\mathbf{y}) \tag{2.7}$$

Here and below

$$U_{ij}(\mathbf{x},\mathbf{y})=U_{ij}^*(\mathbf{x}-\mathbf{y}), \quad T_{ij}(\mathbf{x},\mathbf{y},\mathbf{n})=T_{ij}^*(\mathbf{x}-\mathbf{y},\mathbf{n}) \tag{2.8}$$

If F is a cylindrical surface of type S on which $\upsilon_3=0$, the equality takes the classical form

$$\int_F T_{ij}(\mathbf{y},\mathbf{x},\nu(\mathbf{y}))ds(\mathbf{y})=\rho\delta_{ij}H_F^-(\mathbf{x})$$

similar to Gauss's formula for a double-layer potential [3]. The static analogue of the tensor T_{ij} satisfies exactly the same relation [4].

We will show that the tensor T_{ij}^* is a fundamental solution of Eq. (1.5).

It follows from (2.4) and (2.6) that

$$T_{ji}^* = \Gamma_{ij}^* = K_{il}\left(\frac{\partial}{\partial \mathbf{x}}, \mathbf{n}\right) U_{lj}^*(\mathbf{x})$$
$$K_{il}\left(\frac{\partial}{\partial \mathbf{x}}, \mathbf{n}\right) = \lambda n_i \frac{\partial}{\partial x_l} + \mu n_j \left(\delta_{il}\frac{\partial}{\partial x_j} + \delta_{jl}\frac{\partial}{\partial x_i}\right) \tag{2.9}$$

Consequently

$$A_{ij}\left(\frac{\partial}{\partial \mathbf{x}}\right) T_{jk}^* = A_{ij}\left(\frac{\partial}{\partial \mathbf{x}}\right)\Gamma_{kj}^* = A_{ij}\left(\frac{\partial}{\partial \mathbf{x}}\right) K_{kl}\left(\frac{\partial}{\partial \mathbf{x}}, \mathbf{n}\right) U_{lj}^* =$$
$$= K_{kl}\left(\frac{\partial}{\partial \mathbf{x}}, \mathbf{n}\right)\delta_{il}\delta(\mathbf{x}) = K_{ki}\left(\frac{\partial}{\partial \mathbf{x}}, \mathbf{n}\right)\delta(\mathbf{x}) \tag{2.10}$$

Here we have used the symmetry property of Green's tensor (see (2.2))

$$U_{ij}^*(\mathbf{x}) = U_{ji}^*(\mathbf{x})$$

It follows from (2.2), (2.6) and (2.9) that

$$T_{ij}^*(\mathbf{x}, \mathbf{n}) = \frac{\mu}{c_2}\left((2M_1^2 - M_2^2) n_j f_{1,i} + M_2^2\left(\delta_{ij}\frac{\partial f_2}{\partial \mathbf{n}} + n_i f_{2,j}\right) - 2\frac{\partial}{\partial \mathbf{n}}(f_{2ij} - f_{1ij})\right) \tag{2.11}$$

We shall use the tensors introduced in the integral form of (2.1). As with U_{ijk} we shall separate the bulk and shear components of the tensor T_{ij} : $T_{ij} = T_{ij1} + T_{ij2}$, which are easy to write out using (2.11).

3. SUBSONIC LOADS ($c < c_2$)

In this case [1] the tensor U_{ij}^* has removable singularities along the x_3 axis, except for the point $\mathbf{x} = 0$ because as $r \to 0$, $x_3 \neq 0$

$$V_1 - V_2 \sim \frac{r^2(m_1^2 - m_2^2)}{2|x_3|}, \quad V_1^{-1} - V_2^{-1} \sim \frac{r^2(m_1^2 - m_2^2)}{2|x_3|^2} \tag{3.1}$$

From (2.3) it follows that along any ray passing through the point $\mathbf{x} = 0$, as

$$R \to 0 \quad (R = \sqrt{r^2 + x_3^2})$$

$$U_{ij}^*(\mathbf{x}) \sim \frac{1}{R}\psi_{ij}(\theta, \varphi) \tag{3.2}$$

where ψ_{ij} is a bounded function of θ, $\varphi : \cos\varphi = x_1 / r$, $\sin\varphi = x_2 / r$, $\theta = \arccos(x_3 / R)$.

$U_{ij}^*(\mathbf{x})$ has similar asymptotic behaviour when $R \to \infty$. Consequently the singularity of the tensor T_{ij}^* is of the $1/R^2$ type.

These properties of the tensors $U_{ij}^*(\mathbf{x})$ and $T_{ij}^*(\mathbf{x})$ allow us to write the convolution (2.1) in integral form using the notation (2.4), (2.6) and (2.8)

$$\rho u_i H_s^-(\mathbf{x}) = \int_S (U_{ij}(\mathbf{x},\mathbf{y})p_j(\mathbf{y}) - T_{ij}(\mathbf{x},\mathbf{y},\mathbf{n}(\mathbf{y}))u_j(\mathbf{y}))ds(\mathbf{y}) \tag{3.3}$$

For $\mathbf{x} \in S$ the integrands have no singularities and the integrals converge if $\mathbf{y} \in S$, $\exists\varepsilon > 0$

$$p_j(\mathbf{y}) = O(\|\mathbf{y}\|)^{-\varepsilon}, u_j(\mathbf{y}) = O(1) \quad \text{as } \|\mathbf{y}\| \to \infty \tag{3.4}$$

($p_j(\mathbf{y})$, $u_j(\mathbf{y})$ are locally-integrable functions).

The form of the formula is identical to that of Somigliana's formula of static theory of elasticity [2]. It is also valid for $\mathbf{x} \in S$ if one uses the definition of $H_s^-(\mathbf{x})$ in (2.1). However, in this case the integral containing T_{ij} is singular, and has to be interpreted as a principal value. The proof of this fact is based on the antisymmetry properties of T_{ij} with respect to $\mathbf{x}$ and is similar to that in the static theory of elasticity [5].

For $\mathbf{x} \in S$ the formula gives a singular boundary integral equation for solving fundamental boundary-value problems of the theory of elasticity in the case of subsonic travelling loads.

4. SUPERSONIC LOADS ($c < c_1$)

In this case the components of Green's tensor have a weak singularity of order $(x_3 + m_j r)^{-1/2}$ on the conical fronts $x_3 = -m_j r$. Hence the first integral in (3.3), which contains U_{ij}^*, exists. However, the second integral is formal, because higher-order singularities (like $(x_3 + m_j r)^{-3/2}$) of the tensor T_{ij}^* are not integrable on S for any $\mathbf{x}$. One consequently cannot use formula (3.3) in the supersonic case.

We return to (2.1), taking the differentiation outside the convolution

$$\rho u_i(\mathbf{x}) = U_{ij}^* * p_j(\mathbf{x})\delta_S(\mathbf{x}) + (\lambda u_k n_k(\mathbf{x})\delta_S(\mathbf{x}) * U_{im}^*)_{,m} +$$

$$+\mu((u_j n_m + u_m n_j)\delta_S(\mathbf{x}) * U_{ij}^*)_{,m} + U_{ij}^* * G_j H_S^-(\mathbf{x}) \tag{4.1}$$

In more-abbreviated notation using (1.4) we have

$$\rho u_i(\mathbf{x}) = U_{ij}^* * p_j(\mathbf{x})\delta_S(\mathbf{x}) + C_{jmkl}(u_k r_l(\mathbf{x})\delta_S(\mathbf{x}) * U_{ij}^*)_{,m} + U_{ij}^* * G_j H_S^-(\mathbf{x}) \tag{4.2}$$

The formula can then be written in the integral form

$$\rho u_i H_S^-(\mathbf{x}) = \int_S U_{ij}(\mathbf{x},\mathbf{y})p_i(\mathbf{y})ds(\mathbf{y}) + C_{jmkl}\frac{\partial}{\partial x_m}\int_S U_{ij}(\mathbf{x},\mathbf{y})\, u_k(\mathbf{y})n_l(\mathbf{y})ds(\mathbf{y}) \tag{4.3}$$

where all integrals exist; here $G_j = 0$.

Suppose that the support of the travelling load $\mathbf{p}(\mathbf{y})$ is contained in the set $\mathbf{y} \in S$, $y_3 < 0$, i.e.

$$p_j(\mathbf{y}) = 0 \quad (j = 1,2,3) \quad \text{for } y_3 > 0 \tag{4.4}$$

which corresponds to the physical concept of a real load, which is, as a rule, finite.

Obviously the displacements satisfy the condition that the load overtakes the propagation of perturbations in the medium.

We put $\mathbf{z}_x = (x_1, x_2)\mathbf{z}_y = (y_1; y_2)$, $r = \|\mathbf{z}_x - \mathbf{z}_y\|$ $K_q(\mathbf{x}) = \{\mathbf{y} : y_3 = x_3 + m_q r\}$ are wave fronts of the tensor $U_{ij}(\mathbf{x}, \mathbf{y})$ with vertex at the point $\mathbf{x}$; the interior of these cones is the support of U_{ijq}.

Since in front of $q - m$ fronts $U_{ijq} = 0$ and relation (1.4) holds, (4.3) acquires the form

$$\rho u_i H_S^-(\mathbf{x}) = \sum_{q=1}^{2} \int_D H(-m_q r - x_3) ds(\mathbf{z_y}) \int_{x_3+m_q r}^{0} U_{ijq}(\mathbf{x},\mathbf{y}) \times$$

$$\times p_j(\mathbf{y}) dy_3 + C_{jmkl} \frac{\partial}{\partial x_m} \int_D H(-m_q r - x_3) ds(\mathbf{z_y}) \int_{x_3+m_q r} U_{ijq}(\mathbf{x},\mathbf{y}) u_k(\mathbf{y}) n_l(\mathbf{z_y}) dy_3 \qquad (4.5)$$

Here the integral over the cylindrical surface S is represented in the form of a double integral along the generator and along the directrix D of the cylinder S perpendicular to it; $ds(\mathbf{z_y})$ is the differential of the arc length along D.

We introduce the tensors

$$W_{ij}(\mathbf{x},\mathbf{z_y}) = W_{ij1}(\mathbf{x},\mathbf{z_y}) + W_{ij2}(\mathbf{x},\mathbf{z_y}) \qquad (4.6)$$

$$W_{ijq}(\mathbf{x},\mathbf{z_y}) = H(-x_3 - m_q r) \int_{x_3+m_q r}^{0} U_{ijq}(\mathbf{x},\mathbf{y}) dy_3$$

By calculation we obtain

$$2\pi c^2 W_{ij} = r^{-2}(H_1 V_1^- - H_2 V_2^-)(|x_3|(\delta_{ij}\delta_{[i]3} - r_{,i}r_{,j}) - \delta_{i3}r_{,j} - \delta_{j3}r_{,i}) - H_1 \ln\left(\frac{x_3 + V_1^-}{m_1 r}\right) \times$$

$$\times\left(\delta_{i3}\delta_{j3} + \frac{1}{2} m_1^2 \delta_{ij} \vartheta_{[i]3}\right) + H_2 \ln\left(\frac{x_3 + V_1^-}{m_2 r}\right)\left(\delta_{i3}\delta_{j3} + m_2^2 \delta_{ij}\left(1 + \frac{1}{2}\vartheta_{[i]3}\right)\right) \qquad (4.7)$$

$$H_q = H(-m_q r - x_3)$$

It follows from (4.6) that

$$W_{ijq}(\mathbf{x},\mathbf{z_y}) = 0 \quad \text{for} \quad x_3 > -m_q r \qquad (4.8)$$

In particular

$$W_{ijq}(\mathbf{x},\mathbf{z_y}) = 0 \quad \text{for} \quad x_3 > 0 \qquad (4.9)$$

It is clear that the tensor W_{ij} has only finite discontinuities at the fronts

$$S_q(\mathbf{x}) = \{\mathbf{z_y} : m_q \|\mathbf{z_x} - \mathbf{z_y}\| + x_3 = 0\}.$$

We denote the support of W_{ijq} by $S_q^-(\mathbf{x}) = \{\mathbf{z_y} : m_q \|\mathbf{z_x} - \mathbf{z_y}\| + x_3 < 0\}$. Since when $r \to 0$, $x_3 \neq 0$

$$\frac{1}{r^2}(H_1 V_1^- - H_2 V_2^-) \sim \frac{m_2^2 - m_1^2}{2|x_3|}$$

the tensor $W_{ij}(\mathbf{x}, \mathbf{z_y})$ has no strong singularities with respect to r, and the singularity as $r \to 0$ is of order $\ln r$.

We put $d_q = (y_1, y_2, x_3 + m_q r)$ and perform a near-front regularization of the integrands in relations (4.5)

$$\rho u_i H_S^-(\mathbf{x}) = \sum_{q=1}^{2} \int_D H(-m_q r - x_3) ds(\mathbf{z_y}) \int_{x_3+m_q r}^{0} U_{ijq}(\mathbf{x},\mathbf{y}) p_j(\mathbf{y}) dy_3 +$$

$$+ C_{jmkl} \frac{\partial}{\partial x_m} \int_D H(-m_q r - x_3) n_l(\mathbf{z_y}) ds(\mathbf{z_y}) \int_{x_3+m_q r}^{0} U_{ijq}(\mathbf{x},\mathbf{y})(u_k(\mathbf{y}) - u_k(\mathbf{d_q})) dy_3 +$$

$$+C_{jmkl}\frac{\partial}{\partial x_m}\int_D n_l(\mathbf{z_y})W_{ijq}(\mathbf{x},\mathbf{z_y})u_k(\mathbf{d_q})ds(\mathbf{z_y}) \tag{4.10}$$

Since when $y_3 \to x_3 + m_q r + 0$

$$\frac{u_k(\mathbf{y}) - u_k(\mathbf{d}_q)}{V_q^-(\mathbf{x},\mathbf{y})} \sim \left.\frac{\partial u_k}{\partial y_k}\right|_{\mathbf{y}=\mathbf{d}_q}\sqrt{\frac{y_3 - x_3 - m_q r}{y_3 - x_3 + m_q r}} \to 0$$

the expression under the second integral sign has no singularities at the fronts $K_q(\mathbf{x})$. We also note that S_q is identical with D only if

$$-x_3 > m_q \max_{\mathbf{y}\in S} \|\mathbf{x} - \mathbf{y}\|$$

Consequently, if the inequality is not satisfied, at the ends of the arc of integration S_q we have $x_3 + m_q r = 0$. However, at these points the integrands in the second and third integrals of (4.10) vanish by definition.

These properties allow us to take the differentiation under the integral sign. Here the second integral can be differentiated with respect to all $\mathbf{x}$, and the third only when $\mathbf{x} \notin S$.

It is convenient to introduce tensors H_{ijq} generated from the tensors W_{ijq} and similar to the T_{ijq}

$$H_{jkq}(\mathbf{x},\mathbf{z_y},\mathbf{n}) = C_{imkl}n_l\frac{\partial}{\partial x_m}W_{ijq}(\mathbf{x},\mathbf{z_y}) \tag{4.11}$$

For an isotropic medium they have the same form as (2.11), where W_{lkq} is substituted in place of U_{lk}^*.

Performing the differentiations, we obtain

$$\rho u_i H_S^-(\mathbf{x}) = \sum_{q=1}^{2}\int_D H(-m_q r - x_3)ds(\mathbf{z_y})\times$$

$$\times\int_{x_3+m_q r}^{0}(U_{ijq}(\mathbf{x},\mathbf{y})p_j(\mathbf{y}) - T_{ijq}(\mathbf{x},\mathbf{y},\mathbf{n}(\mathbf{y}))(u_j(\mathbf{y}) - u_j(\mathbf{d}_q)))dy_3 +$$

$$+\int_D H_{ijq}(\mathbf{x},\mathbf{z_y},\mathbf{n}(\mathbf{z_y}))u_j(\mathbf{d}_q)ds(\mathbf{z_y}) \tag{4.12}$$

The formula is similar to Somigliana's formula for supersonic loads. It enables one to find the displacements inside the domain S^- from specified values of the displacements and stresses at the boundary S.

For $x \in S$ the last integral is a singular contour integral. One can show using the definition of $H_S^-(x)$ that equality (2.2) holds if this integral is interpreted in the principal-value sense. In this case (4.11) gives a boundary integral equation for solving the first or second boundary-value problems of elastodynamics for supersonic travelling loads.

5. TRANSONIC LOADS $c_2 < c < c_1$

The above arguments allow one to write down directly a regular integral analogue of the Somigliana formula for transonic velocities

$$\rho u_i H_S^-(\mathbf{x}) = \int_D H(-m_2 r - x_3)ds(\mathbf{z_y})\int_{x_3+m_q r}^{0}(U_{ij2}(\mathbf{x},\mathbf{y})p_j(\mathbf{y}) -$$

$$-T_{ij2}(\mathbf{x},\mathbf{y},\mathbf{n}(\mathbf{y}))(u_j(\mathbf{y}) - u_j(\mathbf{d}_2)))dy_3 + \int_D H_{ij2}(\mathbf{x},\mathbf{z_y},\mathbf{n}(\mathbf{z_y}))u_j(\mathbf{d}_2)ds(\mathbf{z_y}) +$$

$$+\int_D ds(\mathbf{z}_y)\int_{-\infty}^{\infty}(U_{ij1}(\mathbf{x},\mathbf{y})p_j(\mathbf{y})-T_{ij1}(\mathbf{x},\mathbf{y},\mathbf{n}(\mathbf{y}))u_j(\mathbf{y}))dy_3$$

which for $x\in S$ also gives a boundary integral equation for solving the corresponding boundary-value problems.

For sonic velocities when $c=c_2$ Somigliana's formula has a similar form. For $c=c_1$ one should use formula (4.12). In both cases only the form of the tensors U_{ij} and T_{ij} changes.

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Translated by R.L.Z.